

SPECIAL ISOTHERMIC SURFACES OF TYPE  $d$ 

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ABSTRACT. The special isothermic surfaces, discovered by Darboux in connection with deformations of quadrics, admit a simple explanation via the gauge-theoretic approach to isothermic surfaces. We find that they fit into a heirarchy of special classes of isothermic surface and extend the theory to arbitrary codimension.

## INTRODUCTION

Isothermic surfaces, that is, surfaces which admit conformal curvature line coordinates away from umbilics, were the focus of intense activity by the geometers of the turn of the last century [1, 2, 7, 8, 12]. In recent times the subject has attracted new interest initiated by Cieřliński–Goldstein–Sym [10] who pointed out the relationship with integrable systems. Consequently, a highly developed modern theory of isothermic surfaces has emerged which *inter alia* extends the classical theory to arbitrary codimension and more exotic target spaces [3, 4, 18, 17, 15, 21].

However, a substantial part of the motivation and focus of the classical geometers such as Bianchi and Darboux was on a particular class of isothermic surfaces, the *special isothermic surfaces*, which arise from the study of surfaces which are isometric to a quadric. These were isothermic surfaces in  $\mathbb{R}^3$  or, more generally, a 3-dimensional space-form of section curvature  $K$  characterised by the following rather unpromising equation on the principal curvatures:

$$e^{2\theta}(H_u^2 + H_v^2) + \frac{1}{4}M^2 + AM - 2BH + CL + D + \frac{1}{4}L^2K = 0.$$

Here, the first and second fundamental forms of the surface are respectively

$$I = e^{2\theta}(du^2 + dv^2) \text{ and } II = e^{2\theta}(k_1 du^2 + k_2 dv^2),$$

$H = \frac{k_1+k_2}{2}$  is the mean curvature,  $L = e^{2\theta}(k_1 - k_2)$ ,  $M = -HL$  while  $A, B, C, D$  are real constants.

The geometry from which this condition arises is as follows. An isothermic surface  $F$  admits Darboux transforms depending on an initial condition and a spectral parameter  $m \in \mathbb{R}^\times$ : these are new isothermic surfaces which geometrically are the second envelopes of certain sphere congruences enveloping  $F$  and analytically are solutions of a completely integrable system of linear differential equations (a parallel section for a flat connection). Each such gives rise to a congruence of circles which cut the surface and its transform orthogonally at corresponding points and, through each such circle, there is, generically, a unique plane in  $\mathbb{R}^3$ —this is the *circle-plane of the transform*. Darboux [12] proves that an isothermic surface is special of class  $(A, B, C, D)$  if and only if it admits (possibly complex conjugate) Darboux transforms with distinct spectral parameters for which the circle-planes coincide. In this case, there are three such transforms, the *complementary surfaces*, which can be constructed algebraically without integrations and whose spectral parameters are the roots of the cubic equation

$$(1) \quad (m - A)^2 m - Dm + 2BC = 0.$$

Finally, the enveloping surface of this common congruence of circle-planes is isometric to a quadric whose coefficients depend on  $(A, B, C, D)$  also.

Our purpose in this paper is to show that these special isothermic surfaces have a simple explanation in terms of the integrable systems approach to isothermic surfaces. As fruits of this analysis, we shall generalise the classical theory to arbitrary codimension and see that the special isothermic surfaces are a particular case of a hierarchy of natural classes of isothermic surfaces filtered by an integer  $d$ . In codimension 1, the first three of these classes are the maps to  $S^2$  ( $d = 0$ ); the surfaces of constant mean curvature ( $d = 1$ ) and the special isothermic surfaces described above ( $d = 2$ ).

Here is the basic idea: an isothermic surface is characterised by the existence of a pencil of flat metric connections  $\nabla^t$  on a trivial  $\mathbb{R}^{n+1,1}$ -bundle. In this formulation, Darboux transforms with parameter  $m$  are  $\nabla^m$ -parallel null line subbundles. Following an idea of the first author and Calderbank [5], we say that an isothermic surface is *special of type  $d$*  if there is a family  $p(t)$  of  $\nabla^t$ -parallel sections whose dependence on  $t$  is polynomial of degree  $d$ .

It is already proved in [5] that special isothermic surfaces of type 1 are generalised  $H$ -spaces in a space-form and, in section 2.2, we prove that, modulo degeneracies, the special isothermic surfaces of type 2 in codimension 1 are precisely the special isothermic surfaces of Bianchi and Darboux.

In this setting, the complementary surfaces are easy to understand: since the connections  $\nabla^t$  are metric, the polynomial  $(p(t), p(t))$  has constant coefficients (in the classical case, this is the polynomial (1)) and the values  $p(m)$ , at roots  $m$ , span the complementary surfaces.

Moreover, the behaviour of special isothermic surfaces under the various transforms (Darboux, Christoffel, Calapso) of the theory is straightforward to deduce. In all cases, one has explicit gauge transformations, with simple dependence on  $t$ , relating the pencils of flat connections of the original surface and the transform and so can readily construct new families of parallel sections. In this way, we efficiently extend results of Bianchi and Calapso to our entire hierarchy of special isothermic surfaces in arbitrary codimension. All this is the content of section 3.

Finally, in section 4, we return to the roots of the subject and show that Darboux's characterisation via circle-planes extends, *mutatis mutandis*, to special isothermic surfaces of type 2 in any space-form and that, moreover, the special isothermic surfaces of type 2 in  $\mathbb{R}^n$  give rise to surfaces isometric to quadrics.

## 1. PRELIMINARIES

### 1.1. Isothermic and special isothermic surfaces.

**Definition 1.1.** Let  $f : \Sigma \rightarrow E$  be an immersion of a surface  $\Sigma$  in an  $n$ -dimensional Riemannian manifold  $E$ , with  $n \geq 3$ .  $f$  is called an *isothermic surface* if there exist, away from umbilic points, coordinates  $(u, v)$  of  $\Sigma$  such that the first fundamental form  $I := (df, df)$  can be written as

$$I = e^{2\theta}(du^2 + dv^2),$$

for some smooth real function  $\theta$ , and such that  $\frac{\partial}{\partial u}$  and  $\frac{\partial}{\partial v}$  diagonalise simultaneously all shape operators.

In this case  $(u, v)$  are called *conformal curvature line coordinates* of  $f$ . More invariantly, a surface  $f$  is isothermic if there is a holomorphic quadratic differential  $q$  which commutes with the second fundamental form of  $f$ : away from zeros of  $q$ ,

one can find a holomorphic coordinate  $z$  for which  $q^{2,0} = dz^2$  and then  $z = u + iv$  provides the conformal curvature line coordinates.

As we shall see, the class of isothermic surfaces in the  $n$ -sphere  $S^n$  is Möbius invariant, thus preserved by conformal diffeomorphisms.

Quadrics, surfaces of revolution, cones and cylinders in  $\mathbb{R}^3$  are examples of isothermic surfaces as are constant mean curvature surfaces in any 3-dimensional space form.

The following definition is due to Darboux [11], and then developed by Bianchi [1, 2].

**Definition 1.2.** Let  $f : \Sigma \rightarrow E$  be an isothermic surface in a 3-dimensional space-form  $E$  with (sectional) curvature  $K$ . Consider conformal curvature line coordinates  $(u, v)$  of  $f$  with conformal factor  $e^{2\theta}$  and principal curvatures  $k_1$  and  $k_2$  so that the first and second fundamental forms of  $f$  are given by

$$I = e^{2\theta}(du^2 + dv^2) \text{ and } II = e^{2\theta}(k_1 du^2 + k_2 dv^2).$$

$f$  is called a *special isothermic surface* if there are real constants  $A, B, C$ , and  $D$  such that

$$(2) \quad e^{2\theta}(H_u^2 + H_v^2) + \frac{1}{4}M^2 + AM - 2BH + CL + D + \frac{1}{4}L^2K = 0,$$

where  $H = \frac{k_1 + k_2}{2}$  is the mean curvature of  $f$ ,  $L = e^{2\theta}(k_1 - k_2)$  and  $M = -HL$ .

In this case, we say that  $f$  is a *special isothermic surface of class  $(A, B, C, D)$* , with respect to the coordinates  $(u, v)$ .

Equation (2) will be called the *Darboux-Bianchi condition*.

Constant mean curvature surfaces are examples of special isothermic surfaces (in this case the class is not unique, for a given pair of conformal curvature line coordinates).

## 1.2. Conformal submanifold geometry.

**1.2.1. Conformal geometry of the sphere.** We wish to study isothermic and, in particular, special isothermic surfaces in the  $n$ -sphere from a conformally invariant view-point. For this, we find a convenient setting in Darboux's light-cone model of the conformal  $n$ -sphere. So contemplate the light-cone  $\mathcal{L}$  in the Lorentzian vector space  $\mathbb{R}^{n+1,1}$  and its projectivisation  $\mathbb{P}(\mathcal{L})$ . This last has a conformal structure where representative metrics  $g_\sigma$  arise from never-zero sections  $\sigma$  of the tautological bundle  $\pi : \mathcal{L} \rightarrow \mathbb{P}(\mathcal{L})$  via

$$g_\sigma(X, Y) = (d\sigma(X), d\sigma(Y)).$$

Among these metrics are constant sectional curvature metrics given by conic sections: indeed, for  $w \in \mathbb{R}_\times^{n+1,1}$ , set  $E(w) := \{v \in \mathcal{L} : (v, w) = -1\}$  to obtain an  $n$ -dimensional submanifold of  $\mathbb{R}^{n+1,1}$  with induced metric of constant sectional curvature  $-(w, w)$ . In fact, for  $w$  non-null, orthoprojection onto  $\langle w \rangle^\perp$  induces an isometry between  $E(w)$  and  $\{u \in \langle w \rangle^\perp : (u, u) = -1/(w, w)\}$  while, when  $w$  is null, for any choice of  $v_0 \in E(w)$ , orthoprojection onto  $\langle v_0, w \rangle^\perp$  restricts to an isometry of  $E(w)$ . By construction, the bundle projection  $\pi$  restricts to give a conformal diffeomorphism between  $E(w)$  and an open subset of  $\mathbb{P}(\mathcal{L})$ . In particular, choosing unit time-like  $w$  identifies  $\mathbb{P}(\mathcal{L})$  with the conformal  $n$ -sphere while, for  $w$  null, the diffeomorphism  $\mathbb{P}(\mathcal{L}) \setminus \{\langle w \rangle\} \cong \langle v_0, w \rangle^\perp$  is essentially stereoprojection.

The Möbius group  $\text{Möb}(n)$  of conformal diffeomorphisms of  $S^n$  is now readily identified, thanks to Liouville's Theorem, with an open subgroup of the orthogonal group  $O(n+1, 1)$  acting in the obvious way on  $\mathbb{P}(\mathcal{L})$ . This action is transitive with

parabolic stabilisers and each such stabiliser has abelian nilradical so that  $S^n$  is a *symmetric R space*, c.f. [6].

This view-point gives the conformal geometry of the sphere a projective linear flavour. In particular,  $k$ -dimensional subspheres of  $S^n$  are identified with  $(k+1, 1)$ -planes<sup>1</sup> via

$$V \mapsto \mathbb{P}(\mathcal{L} \cap V) \subset \mathbb{P}(\mathcal{L}).$$

Thus, the set of  $k$ -spheres in  $S^n$  is identified with a Grassmannian and so is a (pseudo-Riemannian) symmetric space for  $O(n+1, 1)$ .

**1.2.2. Submanifolds and sphere congruences.** A map  $f : \Sigma \rightarrow S^n \cong \mathbb{P}(\mathcal{L})$  is the same as a null line subbundle  $\Lambda$  of the trivial bundle  $\underline{\mathbb{R}}^{n+1,1} = \Sigma \times \mathbb{R}^{n+1,1}$  via

$$f(x) = \Lambda_x,$$

for all  $x \in \Sigma$ . Given such a  $\Lambda$ , we define

$$\Lambda^{(1)} := \langle F, dF(T\Sigma) \rangle,$$

where  $F$  is any lift (never vanishing section) of  $\Lambda$ . Then  $\Lambda$  is an immersion if and only if  $\Lambda^{(1)}$  is a subbundle of  $\underline{\mathbb{R}}^{n+1,1}$  of rank  $\dim \Sigma + 1$ .

**Definition 1.3.** A *k-sphere congruence* is a map of  $\Sigma$  into the space of  $k$ -spheres or, equivalently, a subbundle  $V$  of  $\underline{\mathbb{R}}^{n+1,1}$  with fibres of signature  $(k+1, 1)$ .

A sphere congruence  $V$  is *enveloped by*  $\Lambda$  if  $\Lambda^{(1)} \subset V$ .

A sphere congruence  $V$  is *orthogonal* to  $\Lambda$  if  $\Lambda \subset V$  and  $\Lambda^{(1)} \subset \Lambda \oplus V^\perp$ .

Here is the geometry: a sphere congruence  $V$  is enveloped by  $\Lambda$  if each sphere  $\mathbb{P}(\mathcal{L} \cap V_x)$  is tangent to the submanifold  $\Lambda$  at  $x$ , for all  $x \in \Sigma$ . Similarly,  $V$  is orthogonal to  $\Lambda$  if the corresponding  $k$ -sphere meets the submanifold with orthogonal tangent planes at each  $x \in \Sigma$ .

There are many sphere congruences enveloped by an immersion  $\Lambda$  but a canonical choice is afforded by the *central sphere congruence* [14] defined as follows:

$$V_{csc} := \langle F, dF(T\Sigma), \Delta F \rangle = \Lambda^{(1)} \oplus \langle \Delta F \rangle,$$

where  $F$  is a lift of  $\Lambda$ , and  $\Delta F$  is the Laplacian of  $F : \Sigma \rightarrow \mathbb{R}^{n+1,1}$ , with respect to the metric  $(dF, dF)$ . This is easily seen to be independent of choices.

### 1.3. Isothermic surfaces and their transformations.

**1.3.1. Gauge theoretic formulation.** There is a gauge-theoretic formulation of the isothermic surface condition [6] that will be basic in all that follows. For this, we begin by recalling the isomorphism  $\bigwedge^2 \mathbb{R}^{n+1,1} \cong \mathfrak{o}(\mathbb{R}^{n+1,1})$  given by

$$(u \wedge v)w = (u, w)v - (v, w)u,$$

for  $u, v, w \in \mathbb{R}^{n+1,1}$ . A map  $\Lambda$  gives rise to a subbundle of abelian subalgebras  $\Lambda \wedge \Lambda^\perp$  of the trivial Lie algebra bundle  $\mathfrak{o}(\underline{\mathbb{R}}^{n+1,1})$ .

**Proposition 1.4** ([6, §3.2.1][16, Lemma 8.6.12]). *An immersion  $\Lambda : \Sigma \rightarrow S^n \cong \mathbb{P}(\mathcal{L})$  is isothermic if and only if there is a non-zero closed 1-form  $\eta \in \Omega^1 \otimes \mathfrak{o}(\mathbb{R}^{n+1,1})$  taking values in  $\Lambda \wedge \Lambda^\perp$ .*

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<sup>1</sup>Thus  $(k+2)$ -dimensional linear subspaces of  $\mathbb{R}^{n+1,1}$  with induced inner product of signature  $(k+1, 1)$ .

The classical and gauge-theoretic view-points are related as follows: given  $\eta \in \Omega^1(\Lambda \wedge \Lambda^\perp)$ , we define a 2-tensor  $q$  on  $\Sigma$  by

$$\frac{1}{2}q(X, Y)F = \eta_X dF(Y),$$

for any lift  $F$  of  $\Lambda$ . This is well-defined and independent of choices since  $\eta\Lambda$  vanishes while  $\eta\Lambda^\perp$  takes values in  $\Lambda$ . Then  $\eta$  is closed if and only if  $q$  is a holomorphic quadratic differential which commutes with the trace-free second fundamental form of  $\Lambda$  [5]. Moreover, in this case,  $\eta$  takes values in  $\Lambda \wedge \Lambda^{(1)}$  and is uniquely determined by  $q$ . Indeed, fix a lift  $F$  and let  $Q$  be the symmetric trace-free endomorphism of  $T\Sigma$  for which

$$\frac{1}{2}q(X, Y) = (dF(QX), dF(Y)).$$

Then

$$(3) \quad \eta = F \wedge dF \circ Q.$$

In particular, if  $z = u + iv$  are conformal curvature line coordinates with  $q^{2,0} = dz^2$  and  $(dF, dF) = e^{2\theta}(du^2 + dv^2)$  then

$$(4) \quad \eta = e^{-2\theta} F \wedge (-F_u du + F_v dv).$$

To summarise:

**Proposition 1.5.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . Then*

- (1)  $\eta \in \Omega^1(\Lambda \wedge \Lambda^{(1)})$ ;
- (2) *For any lift  $F$  of  $\Lambda$ , there is a symmetric, trace-free  $Q \in \Gamma(\text{End}(T\Sigma))$  for which  $\eta = F \wedge (dF \circ Q)$ ;*
- (3)  $\eta$  vanishes on at most a discrete set.

This formulation of the isothermic condition is manifestly conformally invariant: for  $T \in \text{O}(\mathbb{R}^{n+1,1})$  and  $(\Lambda, \eta)$  isothermic, it is clear that  $(T\Lambda, \text{Ad}(T)\eta)$  is isothermic also.

Like the holomorphic quadratic differential to which it is equivalent, the 1-form  $\eta$  is defined up to a real constant scale. However, unless  $\Lambda$  takes values in a 2-sphere, this is the only ambiguity:  $q$  is fixed (up to scaling by a real function) by the requirement that it commute with the trace-free second fundamental form of  $\Lambda$ , unless the latter vanishes. The holomorphicity of  $q$  then forces that scale to be constant. Thus:

**Proposition 1.6** ([4, Proposition 2.4]). *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . Then  $\eta$  is unique up to (non-zero) real scale if and only if  $\Lambda$  is not contained in any 2-sphere.*

A fundamental observation on which the rich transformation theory of isothermic surfaces rests is that setting  $(\nabla^t := d + t\eta)_{t \in \mathbb{R}}$  yields a pencil of flat metric connections on  $\underline{\mathbb{R}}^{n+1,1}$ . Indeed, since  $\Lambda \wedge \Lambda^\perp$  is abelian, the curvature of  $\nabla^t$  is  $t d\eta$  so that, for a non-zero 1-form  $\eta \in \Omega^1(\Lambda \wedge \Lambda^\perp)$ ,  $(\Lambda, \eta)$  is isothermic if and only if  $(\nabla^t)_{t \in \mathbb{R}}$  is a family of flat connections. We now briefly recall how these connections provide transforms of isothermic surfaces.

**1.3.2. Darboux transforms.** Let  $(\Lambda, \eta)$  be an isothermic surface with pencil of flat connections  $\nabla^t$ . Darboux transforms of  $\Lambda$  are spanned by null  $\nabla^m$ -parallel sections of  $\underline{\mathbb{R}}^{n+1,1}$ . More precisely:

**Definition 1.7.** An immersed surface  $\hat{\Lambda} : \Sigma \rightarrow \mathbb{P}(\mathcal{L})$  is a *Darboux transform* with parameter  $m$ ,  $m \in \mathbb{R}^\times$ , of  $\Lambda$  if

- (1)  $\hat{\Lambda} \cap \Lambda = \{0\}$ ;
- (2)  $\hat{\Lambda}$  is  $(d + m\eta)$ -parallel.

The point here is that a Darboux transform  $\hat{\Lambda}$  is also isothermic and there is a simple explicit gauge transformation relating the two pencils of flat connections. For this, we define a family of orthogonal gauge transformations  $\Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(c)$ ,  $c \in \mathbb{R}^\times$ , of  $\underline{\mathbb{R}}^{n+1,1}$  by

$$(5) \quad \Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(c) = \begin{cases} c & \text{on } \hat{\Lambda}, \\ 1 & \text{on } (\Lambda \oplus \hat{\Lambda})^\perp, \\ c^{-1} & \text{on } \Lambda. \end{cases}$$

We now have:

**Proposition 1.8** ([6, Theorem 3.10]). *Let  $\hat{\Lambda}$  be a Darboux transform with parameter  $m$  of an isothermic surface  $(\Lambda, \eta)$ . Then  $\hat{\Lambda}$  is isothermic and the 1-form  $\hat{\eta} \in \Omega^1(\hat{\Lambda} \wedge \hat{\Lambda}^\perp)$  can be chosen so that:*

- (1)  $\Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - \frac{t}{m}) \cdot (d + t\eta) = d + t\hat{\eta}$ , for all  $t \neq m$ ;
- (2)  $(\Lambda, \eta)$  is a Darboux transform with parameter  $m$  of  $(\hat{\Lambda}, \hat{\eta})$ .

Here the action of the gauge transformation on connections is the usual left action:  $\Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - \frac{t}{m}) \cdot (d + t\eta) = \Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - \frac{t}{m}) \circ (d + t\eta) \circ \Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - \frac{t}{m})^{-1}$ .

**Remark 1.9.** The prescription of  $\hat{\eta}$  in Proposition 1.8 amounts to the demand that the two holomorphic quadratic differentials coincide:  $q = \hat{q}$ .

**1.3.3. Christoffel transforms.** Fix a pair  $v_\infty, v_0 \in \mathcal{L}$  with  $\langle v_0, v_\infty \rangle = -1$  so that  $v_0 \in E(v_\infty)$  and conversely. Set  $\mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$ . Orthoprojection is an isometry  $E(v_\infty) \rightarrow \mathbb{R}^n$  with inverse given by

$$x \mapsto \exp(x \wedge v_\infty)v_0 = v_0 + x + \frac{1}{2}(x, x)v_\infty.$$

Now let  $(\Lambda, \eta)$  be isothermic with lift  $F = \exp(f \wedge v_\infty)v_0 : \Sigma \rightarrow E(v_\infty)$ . Then we can write

$$\eta = \text{Ad}(\exp(f \wedge v_\infty))\omega,$$

for  $\omega \in \Omega^1(\langle v_0 \rangle \wedge \langle v_0 \rangle^\perp)$ . It follows from  $d\eta = 0$  that  $d\omega = 0$  [6, Proposition 3.3] so that locally we have  $\omega = df^c \wedge v_0$ , for a map  $f^c : \Sigma \rightarrow \mathbb{R}^n$  defined up to a translation.

We now swop the roles of  $f$  and  $f^c$ ,  $v_0$  and  $v_\infty$  to make:

**Definition 1.10.** The *Christoffel transforms* of  $(\Lambda, \eta)$ , with respect to the pair  $(v_\infty, v_0)$ , are the surfaces  $(\Lambda^c := \langle F^c \rangle, \eta^c)$ , where

$$\begin{aligned} F^c &= \exp(f^c \wedge v_0)v_\infty : \Sigma \rightarrow E(v_0) \\ \eta^c &= \text{Ad}(\exp(f^c \wedge v_0))(df \wedge v_\infty). \end{aligned}$$

We have:

**Proposition 1.11** ([6, Lemma 3.13]). *Let  $(\Lambda^c, \eta^c)$  be a Christoffel transform of  $(\Lambda, \eta)$ . Then*

- (1)  $(\Lambda^c, \eta^c)$  is an isothermic surface;
- (2) For all  $t \in \mathbb{R}^\times$ ,  $\Gamma^c(t) \cdot (d + t\eta) = d + t\eta^c$ , where the gauge transformation  $\Gamma^c(t)$  is given by

$$\Gamma^c(t) = \exp(f^c \wedge v_0) \circ \Gamma_{\langle v_0 \rangle}^{\langle v_\infty \rangle}(t) \circ \exp(-f \wedge v_\infty);$$

- (3)  $(\Lambda, \eta)$  is a Christoffel transform of  $(\Lambda^c, \eta^c)$ .

The geometry behind all this is that the stereoprojection  $f^c$  of a Christoffel transform has parallel tangent planes to those of  $f$  and induces the same conformal structure on  $\Sigma$  as  $f$  while inducing the opposite orientation on  $T\Sigma$  [9, 19].

1.3.4. *T-transforms.* Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . Since each  $\nabla^s = d + s\eta$  is a flat, metric connection on  $\mathbb{R}^{n+1,1}$ , there are local orthogonal gauge transformations  $\Phi_s$ , for each  $s \in \mathbb{R}$ , with  $\Phi_s \cdot \nabla^s = d$ . With this understood, we have:

**Definition 1.12.** The *T-transforms* of  $(\Lambda, \eta)$  are the surfaces  $(\Lambda_s, \eta_s)$ ,  $s \in \mathbb{R}$ , where  $\Lambda_s = \Phi_s \Lambda$  and  $\eta_s = \text{Ad}(\Phi_s)\eta$ .

Note that each  $\Phi_s$ , and so each *T-transform*  $\Lambda_s$ , is defined up to the action of the Möbius group.

We have:

**Proposition 1.13** ([6, §3.3]). *Let  $(\Lambda_s, \eta_s)$  be a T-transform of  $(\Lambda, \eta)$ ,  $s \in \mathbb{R}$ . Then*

- (1)  $(\Lambda_s, \eta_s)$  is an isothermic surface;
- (2) For all  $t \in \mathbb{R}$ ,  $\Phi_s \cdot (d + (t + s)\eta) = d + t\eta_s$ .

This spectral deformation of isothermic surfaces coincides with that introduced, independently, by Calapso [7] and Bianchi [2] for the case  $n = 3$  and extended to higher codimension by Burstall [4] and Schief [21].

## 2. SPECIAL ISOTHERMIC SURFACES OF TYPE $d$

We now come to the central idea of this paper: consider an isothermic surface  $(\Lambda, \eta)$  with its pencil of flat connections  $\nabla^t$ . The theory of ordinary differential equations ensure that we may find  $\nabla^t$ -parallel sections depending smoothly on the spectral parameter  $t$ . However, the existence of such sections with *polynomial* dependence on  $t$  is a condition on  $\Lambda$  of geometric significance. Following Burstall–Calderbank [5], we are therefore led to make the following definition:

**Definition 2.1.** Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$  and let  $p(t) \in \Gamma(\mathbb{R}^{n+1,1})[t]$ . The polynomial  $p(t)$  is called a *polynomial conserved quantity* of  $(\Lambda, \eta)$  if  $p(t)$  is non-zero and  $\nabla^t p(t) \equiv 0$ .

**Proposition 2.2.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . If  $p(t) = \sum_{k=0}^d p_k t^k$  is a polynomial conserved quantity of  $(\Lambda, \eta)$  with degree  $d \in \mathbb{N}_0$ , then*

- (1)  $p_0$  is constant;
- (2)  $p_d$  is a parallel section of  $V_{csc}^\perp$ , with respect to the normal connection;
- (3) the polynomial  $(p(t), p(t)) \in \mathbb{R}[t]$ , that is,  $(p(t), p(t))$  has constant coefficients.

*Proof.* Write  $p(t) = \sum_{k=0}^d p_k t^k$  with  $p_d$  non-zero. Evaluating  $(d + t\eta)p(t)$  at  $t = 0$  we obtain  $dp_0 = 0$ . The top two coefficients of  $\nabla^t p(t) \equiv 0$  read

$$(6a) \quad 0 = \eta p_d$$

$$(6b) \quad 0 = dp_d + \eta p_{d-1},$$

where we have set  $p_{-1} = 0$ . Fix a lift  $F$  of  $\Lambda$ . Proposition 1.5 tells us that we may write  $\eta = F \wedge (dF \circ Q)$  with  $Q \in \Gamma(\text{End}(T\Sigma))$  symmetric, trace-free and bijective off a discrete subset of  $\Sigma$ . Now (6a) reads

$$(F, p_d)dF \circ Q - (dF \circ Q, p_d)F = 0,$$

and, since  $F$  immerses, we see that  $(p_d, F) = 0$  and  $(dF \circ Q, p_d) = 0$  so that  $(dF, p_d) = 0$  off the zero-set of  $Q$ . Hence  $p_d \in \Gamma\Lambda^{(1)\perp}$ .

To see that  $p_d$  takes values in  $V_{csc}$ , it remains to show that  $(d * dF, p_d) = 0$ , where  $*$  is the Hodge star operator on  $\Sigma$ . Now (6b) reads:

$$(7) \quad dp_d + (F, p_{d-1})dF \circ Q - (dF \circ Q, p_{d-1})F = 0,$$

so that

$$\begin{aligned} (d * dF, p_d) &= d(*dF, p_d) - (*dF \wedge dp_d) \\ &= -(F, p_{d-1})(*dF \wedge dF \circ Q) \end{aligned}$$

which last vanishes since  $Q$  is trace-free. Thus,  $p_d \in \Gamma V_{csc}^\perp$ . Moreover, from (7), we see that  $dp_d \in \Omega^1(\Lambda^{(1)}) \subseteq \Omega^1(V_{csc})$ , whence  $\nabla^\perp p_d = 0$ , since the normal connection  $\nabla^\perp$  is just the  $V_{csc}^\perp$ -component of  $d$ .

Finally, since each  $\nabla^t = d + t\eta$  is a metric connection, and  $\nabla^t p(t) \equiv 0$ , we get  $d(p(t), p(t)) = 2(\nabla^t p(t), p(t)) \equiv 0$ .  $\square$

In particular,  $p_d$  is always space-like while,  $p_0$ , if it is non-zero, defines a space-form  $E(p_0)$ .

Observe that if  $n = 3$ , all top terms of polynomial conserved quantities are constant multiples of each other, so that whenever there are two linearly independent polynomial conserved quantities of degree  $d$ , there is necessarily a polynomial conserved quantity of degree  $d - 1$ .

**Definition 2.3.** An isothermic surface  $(\Lambda, \eta)$  in  $S^n$  is a *special isothermic surface of type  $d$*  (with respect to  $p(t)$ ) if it admits a polynomial conserved quantity  $p(t)$  of degree  $d$ .

Moreover, for  $w \in \mathbb{R}_x^{n+1,1}$ , we say that  $(\Lambda, \eta)$  is *special isothermic of type  $d$  in the space-form  $E(w)$*  if  $p_0 \in \langle w \rangle$ .

The cases  $d = 0$  and  $d = 1$  have simple interpretations to which we now turn. Recall that a submanifold of  $S^n$  is *full* if it does not lie in any proper sub-sphere. We have:

**Proposition 2.4.** *An isothermic surface  $(\Lambda, \eta)$  in  $S^n$  is a special isothermic surface of type 0 if and only if  $\Lambda$  is not full.*

*Proof.* Suppose that  $(\Lambda, \eta)$  is a special isothermic surface of type 0, with respect to a polynomial  $p(t) = p_0$ . By Proposition 2.2,  $p_0$  is a constant in  $V_{csc}^\perp$ . Set  $W = \langle p_0 \rangle^\perp$  so that  $W$  is an  $(n, 1)$ -plane with  $\Lambda \subset V_{csc} \subset W$  and  $\Lambda$  has image in the  $(n - 1)$ -sphere  $\mathbb{P}(\mathcal{L} \cap W)$ .

For the converse, if  $\Lambda$  has image in the subsphere  $\mathbb{P}(\mathcal{L} \cap W)$ , for some  $(n, 1)$ -plane  $W = \langle p_0 \rangle^\perp$ , it is easy to see that  $p(t) := p_0$  is a polynomial conserved quantity of  $(\Lambda, \eta)$ .  $\square$

Let us now contemplate the case  $d = 1$ . Recall that a surface  $F$  in a space-form is a *generalised  $H$ -surface* if it admits a parallel unit normal vector field which has constant inner product with the mean curvature vector of  $F$  (see [4, §2.1]). In codimension 1, a generalised  $H$ -surface is simply a surface of constant mean curvature. We now have:

**Proposition 2.5** ([5]). *Let  $(\Lambda, \eta)$  be a full isothermic surface in  $S^n$ , with  $n \geq 3$ . Then  $(\Lambda, \eta)$  is a special isothermic surface of type 1 in the space form  $E(w)$ ,  $w \in \mathbb{R}_x^{n+1,1}$ , if and only if the lift  $F : \Sigma \rightarrow E(w)$  of  $\Lambda$  is a generalised  $H$ -surface.*

*Sketch proof.* Let  $p(t) = p_0 + p_1 t$  be a polynomial conserved quantity of  $(\Lambda, \eta)$ , with  $p_0 \in \langle w \rangle$  non-zero since  $\Lambda$  is full. Without loss of generality, take  $p_1$  to be of unit length. We can then write  $p_1 = (\mathbf{H}, N)F + N$ , for  $\mathbf{H}$  the mean curvature vector of  $F$  and  $N$  a unit parallel section of  $V_w^\perp$ , thus a parallel unit normal to  $F$ . Since  $(p_0, p_1)$  is constant, it immediately follows that  $(\mathbf{H}, N)$  is constant and  $F$  is a generalised  $H$ -surface.



For the converse, if  $F : \Sigma \rightarrow E(w)$  is a generalised  $H$ -surface and  $N$  the corresponding parallel unit normal, set  $p(t) := w + p_1 t$ , where  $p_1 := (\mathbf{H}, N)F + N$ , and  $\eta = F \wedge dp_1$ . One then shows that  $\eta$  is closed and  $(d + t\eta)p(t)$  vanishes identically. For more details, see [5].  $\square$

Note that in codimension 1, the condition that an isothermic surface be special of type  $d = 0, 1$  is a differential equation on the principal curvatures of order  $d$ . This situation persists for all  $d$  [20] and, in particular, we now see that the case  $d = 2$  is very closely related to Darboux's notion formulated in Definition 1.2:

**Theorem 2.6.** *Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^3$ . Fix  $w \in \mathbb{R}_\times^{4,1}$  and consider the lift  $F : \Sigma \rightarrow E(w)$  of  $\Lambda$ . Let  $(u, v)$  be conformal curvature line coordinates corresponding to  $\eta$  so that the first and second fundamental forms of  $F$  are given by*

$$I = e^{2\theta}(du^2 + dv^2) \quad II = e^{2\theta}(k_1 du^2 + k_2 dv^2).$$

*Let  $H = \frac{k_1 + k_2}{2}$  be the mean curvature of  $F$  and set  $L = e^{2\theta}(k_1 - k_2)$  and  $M = -HL$ . Then  $(\Lambda, \eta)$  is special isothermic of degree 2 in  $E(w)$  if and only if there are real constants  $A, B$  and  $C$  such that*

$$(8) \quad \begin{cases} H_{uu} + \theta_u H_u - \theta_v H_v - \frac{1}{2} M k_1 - A k_1 - B e^{-2\theta} + C - \frac{1}{2} L(w, w) = 0 \\ H_{vv} - \theta_u H_u + \theta_v H_v + \frac{1}{2} M k_2 + A k_2 - B e^{-2\theta} - C + \frac{1}{2} L(w, w) = 0. \end{cases}$$

*Proof.* Set  $W_1 := e^{-\theta} F_u$ ,  $W_2 := e^{-\theta} F_v$  and let  $N$  be a unit normal to  $F$  in  $E(w)$ . From (4), we have

$$\eta = e^{-2\theta} F \wedge (-F_u du + F_v dv) = e^{-\theta} F \wedge (-W_1 du + W_2 dv).$$

Now suppose that  $(\Lambda, \eta)$  admits a polynomial conserved quantity  $p(t) = p_0 + p_1 t + p_2 t^2$  with  $p_0 \in \langle w \rangle$ . By Proposition 2.2, we may scale  $p$  by a constant to ensure that  $p_2 = HF + N$ . Denote by  $B$  the constant such that  $p_0 = Bw$  and write

$$p_1 = \alpha F + \beta W_1 + \gamma W_2 + \delta N + \epsilon w,$$

for functions  $\alpha, \beta, \gamma, \delta$  and  $\epsilon$ . With  $p_2 = HF + N$ ,  $dp_2 + \eta p_1 = 0$  is equivalent to

$$\beta = -e^\theta H_u, \quad \gamma = e^\theta H_v, \quad \epsilon = \frac{1}{2} L.$$

Now compute the components of  $dp_1 + \eta p_0$  along the frame  $F, W_1, W_2, N, w$  to conclude that the vanishing of this last is equivalent to

$$(9a) \quad d\alpha = e^{2\theta} H_u(w, w) du + e^{2\theta} H_v(w, w) dv$$

$$(9b) \quad H_{uv} + \theta_u H_v + \theta_v H_u = 0$$

$$(9c) \quad e^{2\theta} dH = \frac{1}{2} (L_u du - L_v dv)$$

$$(9d) \quad d\delta = e^{2\theta} \kappa_1 H_u du - e^{2\theta} \kappa_2 H_v dv$$

$$(9e) \quad H_{uu} + \theta_u H_u - \theta_v H_v + k_1 \delta - \alpha - B e^{-2\theta} = 0$$

$$H_{vv} - \theta_u H_u + \theta_v H_v - k_2 \delta + \alpha - B e^{-2\theta} = 0.$$

Of these, (9b) and (9c) are consequences of the Codazzi equations and, using (9c), (9a) and (9d) can be written

$$d\alpha = \frac{1}{2} (w, w) dL, \quad d\delta = -\frac{1}{2} dM.$$

We therefore introduce constants  $A$  and  $C$  so that  $\delta = -(\frac{1}{2} M + A)$  and  $\alpha = \frac{1}{2} L(w, w) - C$  and conclude that  $p(t)$  is a polynomial conserved quantity if and only

if we have

$$(10a) \quad p_0 = Bw,$$

$$(10b) \quad p_1 = \left(\frac{1}{2}L(w, w) - C\right)F - e^\theta H_u W_1 + e^\theta H_v W_2 - \left(\frac{1}{2}M + A\right)N + \frac{1}{2}Lw$$

$$(10c) \quad p_2 = HF + N$$

and (9e) holds. These equations, which are the components of  $dp_1 + \eta p_0 = 0$  along  $W_1 du$  and  $W_2 dv$  respectively, are precisely (8).  $\square$

Let us relate this to the Bianchi–Darboux condition (2) that characterises the classical special isothermic surfaces. First observe that, with  $p_0, p_1, p_2$  defined by (10), equation (2) reads

$$(p_1, p_1) + 2(p_0, p_2) = D - A^2.$$

Thus a special isothermic surface of type 2 satisfies the Bianchi–Darboux condition, the coefficients of  $(p(t), p(t))$  being constant. Moreover, the converse is almost true as well: if  $\Lambda$  is an isothermic surface satisfying (2), we define  $p(t)$  by (10) and then see that

$$(p(t), p(t)) = t^4 - 2At^3 + (D - A^2)t^2 + 2BCt + B^2(w, w)$$

is constant. Now

$$dp(t) + t\eta p(t) = \omega_1 W_1 du + \omega_2 W_2 dv$$

with  $\omega_1, \omega_2$  determined by (9e) and we have  $0 = d(p(t), p(t)) = 2(dp(t) + t\eta p(t), p(t))$ , the  $t^2$ -coefficient of which is

$$0 = (\omega_1 W_1, p_1)du + (\omega_2 W_2, p_1)dv = e^\theta (-H_u \omega_1 du + H_v \omega_2 dv).$$

We conclude that when  $H_u H_v$  is never zero<sup>2</sup>, a surface is special isothermic of type 2 in a space-form  $E(w)$  if and only if the Bianchi–Darboux condition (2) holds and so is special isothermic in the classical sense.

The class of special isothermic surface of type  $d$  in  $S^n$  is Möbius invariant: if  $\Lambda$  has polynomial conserved quantity  $p(t)$  and  $T \in O(\mathbb{R}^{n+1,1})$  then  $T\Lambda$  has polynomial conserved quantity  $Tp(t)$ . However, the property of being special isothermic in a fixed space-form  $E(w)$  is, of course, not Möbius invariant:  $T\Lambda$  is special isothermic in  $E(Tw)$  and not, as a rule, in  $E(w)$ .

There is an exception to this last rule: if  $p(0) = 0$  then  $\Lambda$  is isothermic of type  $d$  in *any* space-form  $E(w)$ . This is the case if and only if  $q(t) = p(t)/t$  is a polynomial conserved quantity of degree  $d - 1$ . In particular, taking  $n = 3$  and  $d = 2$ , we recover the discussion of Bianchi [1, §§21–22] to conclude that the special isothermic surfaces of type 2 with  $B = 0$  are precisely the images under the Möbius group of constant mean curvature surfaces in some space-form.

### 3. TRANSFORMS OF SPECIAL ISOTHERMIC SURFACES

We now show that the class of special isothermic surfaces is very well-behaved with respect to the transformation theory rehearsed in Section 1.3.

**3.1. Darboux transforms of a special isothermic surface.** Let  $(\Lambda, \eta)$  be an isothermic surface with Darboux transform  $(\hat{\Lambda}, \hat{\eta})$ . From Proposition 1.8, we have

$$\Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - t/m) \cdot (d + t\hat{\eta}) = d + t\hat{\eta}.$$

Thus, if  $p(t)$  is a polynomial conserved quantity for  $(\Lambda, \eta)$ ,  $\Gamma_{\hat{\Lambda}}^{\hat{\Lambda}}(1 - \frac{t}{m})p(t)$  is  $d + t\hat{\eta}$ -parallel and rational in  $t$  with at worst a simple pole at  $m$ .

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<sup>2</sup>This condition is quite strong, excluding, for example, cylinders and surfaces of revolution.

**Theorem 3.1.** *Let  $(\Lambda, \eta)$  be a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}_0$  with respect to a polynomial  $p(t)$ . Let  $\hat{\Lambda}$  be a Darboux transform of  $(\Lambda, \eta)$ , with spectral parameter  $m$ , and let  $G$  be a  $(d + m\eta)$ -parallel lift of  $\hat{\Lambda}$ . Then:*

- 1)  $(p(m), G)$  is constant;
- 2) if  $(p(m), G) = 0$ , then  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$  for which  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ .

*Proof.*  $(p(m), G)$  constant is an immediate consequence of  $p(m)$  and  $G$  being parallel sections with respect to the metric connection  $d + m\eta$ . Assume now that  $p(m)$  is orthogonal to  $\hat{\Lambda}$ . Since  $m$  is a root of the polynomial  $p(t)_\Lambda$ <sup>3</sup>, we can define the polynomial  $\hat{p}(t)$  by  $\hat{p}(t) = \Gamma_\Lambda^\Lambda(1 - \frac{t}{m})p(t)$ , for all  $t \in \mathbb{R} \setminus \{m\}$ , that is, the polynomial  $\hat{p}(t)$  such that

$$p(t)_\Lambda = (1 - \frac{t}{m})\hat{p}(t)_\Lambda, \quad \hat{p}(t)_{\hat{\Lambda}} = (1 - \frac{t}{m})p(t)_{\hat{\Lambda}} \text{ and } \hat{p}(t)_{W^\perp} = p(t)_{W^\perp},$$

which satisfies all the conditions described on point 2. □

In particular, from the previous Theorem, we get a sufficient condition for a Darboux transform of a (classical) special isothermic surface to be again a special isothermic surface of the same class (in a certain 3-dimensional space-form). This classical result can be seen in [1, 2]. This Theorem encodes also a sufficient condition for a Darboux transform of a constant mean curvature surface in a certain space-form to be again a constant mean curvature surface in the same space-form, with the same mean curvature (with the obvious changes, we have the analogous result for generalised H-surfaces).

The next Theorem establishes two results: extends the classical result due to Calapso [8] which claims that in  $\mathbb{R}^3$  all Darboux transforms of a constant mean curvature surface are special isothermic surfaces; guarantees that the sufficient condition of Theorem 3.1 is also a necessary condition, with the assumptions that codimension is 1 and that  $(\Lambda, \eta)$  is not a special isothermic surface of type less than  $d$  (see [1] for the classical result).

**Theorem 3.2.** *Let  $(\Lambda, \eta)$  be a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}_0$  with respect to a polynomial  $p(t)$ , and let  $\hat{\Lambda}$  be a Darboux transform of  $(\Lambda, \eta)$ , with parameter  $m$ . Then:*

- 1)  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d + 1$ , with respect to a polynomial  $\hat{p}(t)$  for which  $\hat{p}(0) = p(0)$ ;
- 2) if  $n = 3$ ,  $d \in \mathbb{N}$ ,  $(\Lambda, \eta)$  is not a special isothermic surface of type  $d - 1$ , and  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$  such that  $\hat{p}(0) = p(0)$ , then  $p(m) \in \Gamma(\hat{\Lambda}^\perp)$ .<sup>4</sup>

*Proof.* Consider the  $(d + 1)$ -th degree polynomial

$$q(t) = (1 - \frac{t}{m})p(t),$$

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<sup>3</sup>For each section  $\varphi$  of  $\mathbb{R}^{n+1,1}$ , write

$$\varphi = \varphi_\Lambda + \varphi_{\hat{\Lambda}} + \varphi_{W^\perp}$$

for the decomposition of  $\varphi$  corresponding to  $\mathbb{R}^{n+1,1} = \Lambda \oplus \hat{\Lambda} \oplus W^\perp$ , where  $W := \Lambda \oplus \hat{\Lambda}$ .

<sup>4</sup>This result is obviously true if we take  $d = 0$ , with arbitrary codimension.

which is a polynomial conserved quantity of  $(\Lambda, \eta)$ . Since  $q(m) = 0$ , and therefore  $q(m) \in \Gamma(\hat{\Lambda}^\perp)$ , we guarantee that  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d+1$ , with respect to a polynomial  $\hat{q}(t)$  such that  $\hat{q}(0) = q(0)$ , and then  $\hat{q}(0) = p(0)$ .

Assume now that  $n = 3$ ,  $d \in \mathbb{N}$  and that  $(\Lambda, \eta)$  is not a special isothermic surface of type  $d-1$ . Suppose that  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$ , such that  $\hat{p}(0) = p(0)$ . Based on the first part of this proof, it follows that the polynomial  $q(t)$  satisfying  $q(t) = \Gamma_{\hat{\Lambda}}^\Lambda(1 - \frac{t}{m})(1 - \frac{t}{m})\hat{p}(t)$  is a polynomial conserved quantity of  $(\Lambda, \eta)$ , with degree  $d+1$ .

Take the polynomial conserved quantity  $p'(t) := (1 - \frac{t}{m})p(t)$  of  $(\Lambda, \eta)$ . Since  $q(0) = p'(0)$ , consider the polynomial  $s(t)$  such that  $ts(t) = q(t) - p'(t)$ . One can prove that  $q(t) - p'(t)$  has degree  $d+1$  or  $q(t) - p'(t)$  is the zero polynomial.

If  $q(t) - p'(t)$  has degree  $d+1$ , we get that  $s(t)$  is a polynomial conserved quantity of  $(\Lambda, \eta)$  with degree  $d$ , and then, taking into account that the codimension is 1, we conclude that there is a real constant  $\alpha$  such that  $p(t) = \alpha s(t)$ . Consequently  $p(m)_\Lambda = 0$ ; if  $q(t) - p'(t) \equiv 0$ , we get  $q(m) = p'(m) = 0$ . Take the polynomial  $a(t)$  of degree  $d$  such that  $q(t) = (1 - \frac{t}{m})a(t)$ . In this case we also have  $p(t) = \alpha a(t)$ , for some real  $\alpha$ , and  $a(m)_\Lambda = 0$  (because  $(1 - \frac{1}{m}t)\hat{p}(t)_\Lambda = a(t)_\Lambda$ ).  $\square$

From this Theorem we obtain in particular examples of special isothermic surfaces of type 2, taking a special isothermic surface of type 1, i.e., a generalised  $H$ -surface in some space-form (which does not live in a 2-sphere) and considering its Darboux transforms.

For a given special isothermic surface with respect to a polynomial  $p(t)$ , there are particular Darboux transforms which are obtained without integration. They arise from the roots of  $(p(t), p(t))$ .

**Definition 3.3.** Let  $(\Lambda, \eta)$  be a special isothermic surface in  $S^n$ , with respect to  $p(t)$ . The Darboux transforms  $\hat{\Lambda}$  of  $(\Lambda, \eta)$  such that  $\hat{\Lambda} = \langle p(m) \rangle$ , for some  $m \in \mathbb{R}^\times$  satisfying  $(p(m), p(m)) = 0$ , are called the *complementary surfaces* of  $(\Lambda, \eta)$ , with respect to  $p(t)$ .

We obtain therefore at most  $2d$  complementary surfaces for each special isothermic surface of type  $d$ .

Note that for  $d = 1$  with  $0 \neq p(0) \in \mathcal{L}$ , we have exactly one complementary surface. Fixing a  $v_0 \in E(p(0))$ , it is straightforward to check that it is the only surface in  $S^n$  such that its projection in  $\mathbb{R}^n = \langle v_0, p(0) \rangle^\perp \cong E(p(0))$  is simultaneously a Darboux transform and a Christoffel transform of the projection  $f$  of  $\Lambda$  in  $\mathbb{R}^n$ . In  $\mathbb{R}^n$  it is given by  $f + \frac{1}{H}N$ , where  $N$  is a unit normal section of  $f$ , parallel with respect to the normal connection such that  $H = (\mathbf{H}, N)$  is constant ( $\mathbf{H}$  is denoting the mean curvature vector of  $f$ ). We get in particular that this complementary surface of  $\Lambda$  is also a generalised  $H$ -surface in the space-form  $E(p(0))$ . This result is a particular case of the following Corollary of Theorem 3.1.

**Corollary 3.4.** Let  $(\Lambda, \eta)$  be a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}$  with respect to a polynomial  $p(t)$ . Let  $\hat{\Lambda}$  be a complementary surface of  $(\Lambda, \eta)$  with respect to  $p(t)$ . Then  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$ , such that  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ .

*Proof.* Denoting by  $m$  the spectral parameter of  $\hat{\Lambda}$ , observe that the lift  $G := p(m)$  of  $\hat{\Lambda}$  is a  $(d + m\eta)$ -parallel section, which is orthogonal to  $\hat{\Lambda}$ .  $\square$

Taking  $d = 2$ , with codimension 1, we obtain the corresponding classical result (see [1, 2]).

To conclude this section, we will present the conditions in which a special isothermic surface of type  $d$  admits Darboux transforms which are special isothermic surfaces of type  $d - 1$ .

**Proposition 3.5.** *Let  $(\Lambda, \eta)$  be a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}$  with respect to  $p(t)$ . Suppose that  $(\Lambda, \eta)$  is not a special isothermic surface of type  $d - 1$ . Then  $(\Lambda, \eta)$  admits a Darboux transform with spectral parameter  $m$ , which is a special isothermic surface of type  $d - 1$  with respect to a polynomial  $\hat{p}(t)$  such that  $\hat{p}(0) = p(0)$  if and only if  $m \in \mathbb{R}^\times$  is a repeated root of  $(p(t), p(t))$ . Furthermore, the Darboux transform in this situation is the complementary surface  $\langle p(m) \rangle$  of  $(\Lambda, \eta)$ .*

*Proof.* Consider a Darboux transform  $\hat{\Lambda}$  of  $(\Lambda, \eta)$ , with parameter  $m$ , and assume that  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d - 1$ , with respect to a polynomial  $\hat{p}(t)$  such that  $\hat{p}(0) = p(0)$ . We know that the polynomial  $\xi(t)$  satisfying  $\xi(t) = \Gamma_{\hat{\Lambda}}^\Lambda(1 - \frac{t}{m})(1 - \frac{t}{m})\hat{p}(t)$  is a polynomial conserved quantity of  $(\Lambda, \eta)$ , with degree  $d$ . Since  $\xi(0) = \hat{p}(0) = p(0)$  and  $p(t)$  and  $\xi(t)$  are polynomials of degree  $d$ , we have necessarily  $p(t) = \xi(t)$  because  $d$  is the least degree of the polynomials which are conserved quantities of  $(\Lambda, \eta)$ . We now get that  $m$  is a repeated root of  $(p(t), p(t))$  because

$$(p(t), p(t)) = (\xi(t), \xi(t)) = (1 - \frac{1}{m}t)^2(\hat{p}(t), \hat{p}(t)).$$

Observe that  $p(m)$  is not the zero section. Indeed, if it was, we would have  $p(t) = (t - m)q(t)$ , for a certain polynomial  $q(t)$  of degree  $d - 1$ . But then  $q(t)$  would be a polynomial conserved quantity of  $(\Lambda, \eta)$ , which is absurd. By virtue of  $p(m)$  being a non-zero  $(d + m\eta)$ -parallel section, we conclude that  $p(m)$  never vanishes. Therefore  $p(m) \in \Gamma(\mathcal{L})$ . Consequently, since  $p(m)_\Lambda = 0$ , we obtain  $\langle p(m) \rangle + \hat{\Lambda} \subset \mathcal{L}$ , and then  $\hat{\Lambda} = \langle p(m) \rangle$ .

Conversely, assume that there is a repeated root  $m \in \mathbb{R}^\times$  of  $(p(t), p(t))$ . We already know, from the first part, that  $p(m)$  never vanishes. Consider the null line subbundle  $\hat{\Lambda} = \langle p(m) \rangle$  of  $\underline{\mathbb{R}}^{n+1,1}$ . Note that we can assume that  $\Lambda \cap \hat{\Lambda} = \{0\}$  considering a non-empty open subset of  $\Sigma$ , if necessary, because if  $p(m) \in \Gamma(\Lambda)$ , we would have  $\eta p(m) = 0$ , which would imply that  $dp(m) = -m\eta p(m) = 0$ . Hence we would get  $\Lambda = \langle p(m) \rangle$  constant, which is absurd. Furthermore, one can prove that  $\langle p(m) \rangle$  is an immersion (considering again a smaller set, if necessary), having in mind Proposition 1.5. We can therefore take the Darboux transform  $\hat{\Lambda} = \langle p(m) \rangle$  of  $(\Lambda, \eta)$  (a complementary surface of  $(\Lambda, \eta)$ , with respect to  $p(t)$ ). Considering the polynomial conserved quantity  $\hat{p}(t)$  of  $(\hat{\Lambda}, \hat{\eta})$  defined by  $\hat{p}(t) = \Gamma_{\hat{\Lambda}}^\Lambda(1 - \frac{t}{m})p(t)$  which has degree  $d$ , and satisfies  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ , we get that  $m$  is a repeated root of the polynomial

$$2(\hat{p}(t)_\Lambda, \hat{p}(t)_{\hat{\Lambda}}) + (\hat{p}(t)_{(\Lambda \oplus \hat{\Lambda})^\perp}, \hat{p}(t)_{(\Lambda \oplus \hat{\Lambda})^\perp}).$$

Since  $\hat{p}(m)_{(\Lambda \oplus \hat{\Lambda})^\perp} = p(m)_{(\Lambda \oplus \hat{\Lambda})^\perp} = 0$ , consider the polynomial  $a(t)$  for which  $\hat{p}(t)_{(\Lambda \oplus \hat{\Lambda})^\perp} = (m - t)a(t)$ . It follows that  $m$  is a repeated root of the polynomial

$$2(1 - \frac{1}{m}t)(\hat{p}(t)_\Lambda, p(t)_{\hat{\Lambda}}) + (m - t)^2(a(t), a(t)),$$

which guarantees that  $m$  is a root of  $(\hat{p}(t)_\Lambda, p(t)_{\hat{\Lambda}})$ . Since  $p(m)_{\hat{\Lambda}}$  is always different from zero, we get that  $\hat{p}(m)_\Lambda = 0$ . Therefore, as a consequence of

$$\hat{p}(m) = \hat{p}(m)_\Lambda + \hat{p}(m)_{\hat{\Lambda}} + \hat{p}(m)_{(\Lambda \oplus \hat{\Lambda})^\perp} = 0,$$

we will consider the polynomial  $s(t)$  of degree  $d - 1$  such that  $\hat{p}(t) = (1 - \frac{1}{m}t)s(t)$ , which will be a polynomial conserved quantity of  $(\hat{\Lambda}, \hat{\eta})$  satisfying  $s(0) = \hat{p}(0) = p(0)$ .  $\square$

With this last Proposition, we proved that given a special isothermic surface  $(\Lambda, \eta)$  of type  $d \in \mathbb{N}$ , with respect to a polynomial  $p(t)$ , which is not a special isothermic surface of type  $d - 1$ , the Darboux transforms  $(\hat{\Lambda}, \hat{\eta})$  of  $(\Lambda, \eta)$  which are special isothermic surfaces of type  $d - 1$  with respect to a polynomial with constant term  $p(0)$  are exactly the complementary surfaces of  $(\Lambda, \eta)$  with respect to  $p(t)$  such that the spectral parameters are repeated (non-zero) roots of  $(p(t), p(t))$ . In particular, we get that there are at most  $d$  Darboux transforms  $(\hat{\Lambda}, \hat{\eta})$  of  $(\Lambda, \eta)$  under these conditions. The case in which 0 is a repeated root of  $(p(t), p(t))$  will be discussed in section 3.2 (see Propositions 3.9 and 3.10).

**3.1.1. Bianchi permutability theorem of special isothermic surfaces.** Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$  and let  $(\Lambda_1, \eta_1)$  and  $(\Lambda_2, \eta_2)$  be two Darboux transforms of  $(\Lambda, \eta)$ , associated to different spectral parameters  $m_1$  and  $m_2$ , respectively. Assume that  $\Lambda_1 \cap \Lambda_2 = \{0\}$ . The Bianchi permutability theorem of isothermic surfaces claims that there is an isothermic surface in  $S^n$  which is simultaneously a Darboux transform of  $(\Lambda_1, \eta_1)$ , with spectral parameter  $m_2$ , and a Darboux transform of  $(\Lambda_2, \eta_2)$ , with parameter  $m_1$ , namely

$$\hat{\Lambda} := \Gamma_{\Lambda}^{\Lambda_1} \left(1 - \frac{m_2}{m_1}\right) (\Lambda_2) = \Gamma_{\Lambda}^{\Lambda_2} \left(1 - \frac{m_1}{m_2}\right) (\Lambda_1) = \Gamma_{\Lambda_2}^{\Lambda_1} \left(\frac{m_2}{m_1}\right) (\Lambda).$$

$(\hat{\Lambda}, \hat{\eta})$  is isothermic when we consider the 1-form

$$\hat{\eta} := \Gamma_{\Lambda_2}^{\Lambda_1} \left(\frac{m_2}{m_1}\right) \circ \eta \circ \Gamma_{\Lambda_2}^{\Lambda_1} \left(\frac{m_2}{m_1}\right)^{-1}.$$

In fact this  $\hat{\eta}$  is the closed 1-form associated to  $\hat{\Lambda}$  as a Darboux transform of  $(\Lambda_1, \eta_1)$  with parameter  $m_2$ , and associated to  $\hat{\Lambda}$  as a Darboux transform of  $(\Lambda_2, \eta_2)$  with parameter  $m_1$ .

With the above notations, we have the following Theorem.

**Theorem 3.6.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}_0$ , with respect to a polynomial  $p(t)$  for which  $p(m_1) \in \Gamma(\Lambda_1^\perp)$  and  $p(m_2) \in \Gamma(\Lambda_2^\perp)$ , then  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$  for which  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ .*

*Proof.* Assume that  $(\Lambda, \eta)$  is a special isothermic surface of type  $d \in \mathbb{N}_0$ , with respect to a polynomial  $p(t)$  for which  $p(m_1) \in \Gamma(\Lambda_1^\perp)$  and  $p(m_2) \in \Gamma(\Lambda_2^\perp)$ . Theorem 3.1 guarantees that the polynomial  $p_1(t)$  such that  $p_1(t) = \Gamma_{\Lambda}^{\Lambda_1} \left(1 - \frac{t}{m_1}\right) p(t)$ , for all  $t \in \mathbb{R} \setminus \{m_1\}$ , is a polynomial conserved quantity of  $(\Lambda_1, \eta_1)$ , with degree  $d$ , satisfying  $p_1(0) = p(0)$  and  $(p_1(t), p_1(t)) = (p(t), p(t))$ . It follows from  $p(m_2) \in \Gamma(\Lambda_2^\perp)$  that

$$p_1(m_2) = \Gamma_{\Lambda}^{\Lambda_1} \left(1 - \frac{m_2}{m_1}\right) p(m_2) \in \left(\Gamma_{\Lambda}^{\Lambda_1} \left(1 - \frac{m_2}{m_1}\right) (\Lambda_2)\right)^\perp = \hat{\Lambda}^\perp,$$

which implies, using again Theorem 3.1, that  $(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$ , namely the one satisfying  $\hat{p}(t) = \Gamma_{\Lambda_1}^{\hat{\Lambda}} \left(1 - \frac{t}{m_2}\right) p_1(t)$ , such that  $\hat{p}(0) = p_1(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p_1(t), p_1(t))$ . Consequently  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ .  $\square$

As a consequence of Theorems 3.6 and 3.2, we obtain the following Corollary. One can find the corresponding classical result in [1].

**Corollary 3.7.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^3$  of type  $d \in \mathbb{N}$ , with respect to a polynomial  $p(t)$ , but it is not a special isothermic surface of type  $d - 1$ , and if  $(\Lambda_1, \eta_1)$  and  $(\Lambda_2, \eta_2)$  are special isothermic surfaces of type  $d$ , with respect to polynomials  $p_1(t)$  and  $p_2(t)$ , respectively, such that  $p_1(0) = p_2(0) = p(0)$ , then*

$(\hat{\Lambda}, \hat{\eta})$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $\hat{p}(t)$  for which  $\hat{p}(0) = p(0)$  and  $(\hat{p}(t), \hat{p}(t)) = (p(t), p(t))$ .

**3.2. Christoffel transforms of a special isothermic surface.** Let  $(\Lambda, \eta)$  be an isothermic surface in  $S^n$ . Fix a pair  $(v_\infty, v_0)$  where  $v_\infty, v_0 \in \mathcal{L}$  and  $(v_0, v_\infty) = -1$ .

**Theorem 3.8.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}_0$  with respect to a polynomial  $p(t)$  for which the constant term  $p(0) \in \langle v_\infty \rangle^\perp$ , then  $(\Lambda^c, \eta^c)$  is a special isothermic surface of type  $d$ , with respect to a polynomial  $q(t)$  such that  $(q(t), q(t)) = (p(t), p(t))$  and  $q(0) \in \langle v_0 \rangle^\perp$ . Moreover, if  $p(0) \in \langle v_\infty \rangle$ , then  $q(0) \in \langle v_0 \rangle$ .*

*Proof.* Assuming that a polynomial  $p(t) \in \Gamma(\mathbb{R}^{n+1,1})[t]$  satisfies  $p(0) \in \langle v_\infty \rangle^\perp$ , we can define the polynomial  $q(t)$  by  $q(t) = \Gamma^c(t)p(t)$ , for all  $t \neq 0$  (recall section 1.3.3). If  $p(t)$  is  $(d + t\eta)$ -parallel, we automatically obtain that  $q(t)$  is  $(d + t\eta^c)$ -parallel. Furthermore,

$$q(0) = -(p_1, v_\infty)v_0 + p_0 + (p_0, v_0)v_\infty + (p_0, f^c)v_0,$$

where  $p(t) = \sum_{k=0}^d p_k t^k$ , and  $f^c : \Sigma \rightarrow \mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$  is an immersion such that  $\eta := \text{Ad}(\exp(f \wedge v_\infty)) (df^c \wedge v_0)$ .  $\square$

The last part of this Theorem gives in particular the classical result, due to Bianchi [1], which states that given a special isothermic surface  $f$  in  $\mathbb{R}^3$  of class  $(A, B, C, D)$ , with conformal curvature line coordinates  $(u, v)$ , the Christoffel transforms  $f^c$  of  $f$  such that

$$f_u^c = e^{-2\theta} f_u \text{ and } f_v^c = -e^{-2\theta} f_v,$$

where the first fundamental form of  $f$  is  $I = e^{2\theta}(du^2 + dv^2)$ , are special isothermic surfaces of class  $(A, C, B, D)$ . Indeed, fix  $v_\infty \in \mathcal{L} \subseteq \mathbb{R}^{4,1}$  and  $v_0 \in E(v_\infty)$ , and identify  $\mathbb{R}^3$  with  $\langle v_0, v_\infty \rangle^\perp$ . Consider the surfaces  $F = \exp(f \wedge v_\infty)v_0$  in  $E(v_\infty)$  and  $F^c = \exp(f^c \wedge v_0)v_\infty$  in  $E(v_0)$ . Take now the isothermic surfaces  $(\Lambda := \langle F \rangle, \eta)$  and  $(\Lambda^c := \langle F^c \rangle, \eta^c)$  where  $\eta = \text{Ad}(\exp(f \wedge v_\infty)) (df^c \wedge v_0)$  and  $\eta^c = \text{Ad}(\exp(f^c \wedge v_0)) (df \wedge v_\infty)$ .

Consider the unit vector fields  $W_1 = e^{-\theta} F_u$  and  $W_2 = e^{-\theta} F_v$ , and let  $N$  be a unit normal to  $F$  in  $E(v_\infty)$ . Take the unit vector fields  $W_1^c := e^\theta F_u^c = \Gamma^c(t)W_1$ ,  $W_2^c := e^\theta F_v^c = -\Gamma^c(t)W_2$  and  $N^c = -\Gamma^c(t)N$  of  $F^c$ .

By virtue of  $F$  being a special isothermic surface of class  $(A, B, C, D)$ , we learn that  $p(t) = p_0 + p_1 t + p_2 t^2$  given by

$$p_0 = Bv_\infty,$$

$$p_1 = -CF - e^\theta H_u W_1 + e^\theta H_v W_2 - \left(\frac{1}{2}M + A\right)N + \frac{1}{2}Lv_\infty \text{ and}$$

$$p_2 = HF + N,$$

is a polynomial conserved quantity of  $(\Lambda, \eta)$  (in fact assuming that  $H_u H_v$  never vanishes). Considering now the polynomial  $q(t) = q_0 + q_1 t + q_2 t^2$  for which  $q(t) = \Gamma^c(t)p(t)$ , for all  $t \neq 0$ , we get

$$q_0 = -Cv_0,$$

$$q_1 = BF^c - e^\theta H_u W_1^c - e^\theta H_v W_2^c + \left(\frac{1}{2}M + A\right)N^c + Hv_0 \text{ and}$$

$$q_2 = \frac{1}{2}LF^c - N^c.$$

The result follows noticing that  $H^c = -\frac{1}{2}L$ ,  $L^c = -2H$  and  $M^c = M$  (with the obvious notations).

**Proposition 3.9.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}$ , with respect to a polynomial  $p(t)$  for which  $0 \neq p(0) \in \langle v_\infty \rangle$  and 0 is a repeated root of  $(p(t), p(t))$ , then  $(\Lambda^c, \eta^c)$  is a special isothermic surface of type  $d - 1$ .*

*Proof.* Assume the conditions of the hypothesis with  $p(t) = \sum_{k=0}^d p_k t^k$ . Take the  $d$ -degree polynomial  $q(t)$  defined by  $\Gamma^c(t)p(t)$ , for all  $t \neq 0$ , which is a conserved quantity of  $(\Lambda^c, \eta^c)$ . Since 0 is a repeated root of  $(p(t), p(t))$ , we get  $(p_0, p_1) = 0$ , which implies that  $(p_1, v_\infty) = 0$ , and then  $q(0) = -(p_1, v_\infty)v_0 = 0$ . Considering the polynomial  $s(t)$  for which  $q(t) = ts(t)$ , we obtain a polynomial conserved quantity of  $(\Lambda^c, \eta^c)$ , with degree equal to  $d - 1$ .  $\square$

**Proposition 3.10.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^3$  of type  $d \in \mathbb{N}$  with respect to a polynomial  $p(t)$ , but it is not a special isothermic surface of type  $d - 1$ , and if  $(\Lambda^c, \eta^c)$  is a special isothermic surface of type  $d - 1$ , then 0 is a repeated root of  $(p(t), p(t))$  and  $p(0) \in \langle v_\infty \rangle$ .*

*Proof.* Assume the conditions of the hypothesis. Denote by  $s(t)$  a polynomial of degree  $d - 1$  which is a polynomial conserved quantity of  $(\Lambda^c, \eta^c)$ . Then the polynomial  $g(t)$  defined by  $g(t) = \Gamma^c(t)^{-1}(ts(t))$ , for all  $t \neq 0$ , is a polynomial conserved quantity of  $(\Lambda, \eta)$ , with degree  $d$ . Using the fact that  $(\Lambda, \eta)$  is not a special isothermic surface of type  $d - 1$  and the fact that the codimension is 1, we obtain  $p(t) = \alpha g(t)$ , for some non-zero constant  $\alpha$ . Hence we conclude that  $(p(t), p(t)) = \alpha^2(g(t), g(t)) = \alpha^2 t^2(s(t), s(t))$ , which guarantees that 0 is a repeated root of  $(p(t), p(t))$ . Finally, we get that  $p(0) \in \langle v_\infty \rangle$ , because the constant term of  $ts(t)$  belongs to  $\langle v_0 \rangle$  (it is equal to zero).  $\square$

**Remark 3.11.** Consider a special isothermic surface  $(\Lambda, \eta)$  in  $S^n$  of type  $d \in \mathbb{N}$ , with respect to a polynomial  $p(t)$ . If  $p_0 := p(0) \in \mathcal{L}$ , we can consider  $p_0$  as the point at infinity  $v_\infty$ , and apply all these results. Note that  $p_0 \in \mathcal{L}$  if and only if  $p_0 \neq 0$  and 0 is a root of  $(p(t), p(t))$ .

Observe that if  $(p(t), p(t))$  admits a repeated root  $m$ , we can therefore conclude, in general, that  $(\Lambda, \eta)$  admits Darboux transform(s) (which will be complementary surfaces) or Christoffel transforms which are special isothermic surfaces of type  $d - 1$ .

### 3.3. T-transforms of a special isothermic surface.

**Theorem 3.12.** *If  $(\Lambda, \eta)$  is a special isothermic surface in  $S^n$  of type  $d \in \mathbb{N}_0$  with respect to a polynomial  $p(t)$ , then  $(\Lambda_s, \eta_s)$  is a special isothermic surface of type  $d$ , with respect to a polynomial with constant term equal to  $\Phi_s(p(s))$ .*

*Proof.* Given a polynomial conserved quantity  $p(t)$  of  $(\Lambda, \eta)$ ,  $p(t+s)$  is  $(d + (t+s)\eta)$ -parallel, which guarantees that  $q(t) := \Phi_s p(t+s)$  is parallel with respect to  $d + t\eta_s = \Phi_s \cdot (d + (t+s)\eta)$ .  $\square$

The corresponding classical result of  $T$ -transforms of a special isothermic surfaces can be found in [2], which states that the  $T$ -transforms of a special isothermic surface are special isothermic surfaces, but in general of a different class and in a different space-form.

Lawson correspondence is a consequence of Theorem 3.12 when we consider the case of special isothermic surfaces in  $S^3$  of type 1, which is related, as we know, to constant mean curvature surfaces in a 3-dimensional space-form. Indeed, consider an arbitrary full isothermic surface  $(\Lambda, \eta)$  in  $S^3$  which is a constant mean curvature surface in a space-form  $E(w)$  (with sectional curvature equal to  $K := -(w, w)$ ). Denote by  $F$  the lift of  $\Lambda$  which lives in  $E(w)$ , by  $N$  a unit normal to  $F$  in  $E(w)$



and finally by  $H$  the (constant) mean curvature of  $F$  with respect to  $N$ . We have that  $p(t) := w + p_1 t$ , with  $p_1 = HF + N$ , is a polynomial conserved quantity of  $(\Lambda, \eta)$ , considering  $\eta = F \wedge dp_1$ .

Given an arbitrary  $T$ -transform  $(\Lambda_s, \eta_s)$  of  $(\Lambda, \eta)$  (associated to a  $\Phi_s$ ), take the polynomial conserved quantity

$$q(t) := \Phi_s p(t + s) = \Phi_s(p(s)) + \Phi_s(p_1)t$$

of  $(\Lambda_s, \eta_s)$ . Therefore we conclude that  $\Lambda_s$  is a constant mean curvature surface in the space-form  $E(\Phi_s(p(s)))$ , which has sectional curvature  $K_s = -(\Phi_s(p(s)), \Phi_s(p(s)))$ , with (constant) mean curvature equal to  $H_s := -(\Phi_s(p_1), \Phi_s(p(s)))$ , with respect to  $\Phi_s(N + sF)$ . Since

$$(q(t), q(t)) = (\Phi_s p(t + s), \Phi_s p(t + s)) = (p(t + s), p(t + s)),$$

we obtain, in particular, that the discriminants of both second degree polynomials

$$(q(t), q(t)) = -K_s - 2H_s t + t^2 \text{ and } (p(t), p(t)) = -K - 2Ht + t^2$$

are equal. Consequently,  $4H_s^2 + 4K_s = 4H^2 + 4K$ , i.e.,

$$H_s^2 + K_s = H^2 + K.$$

#### 4. SPHERICAL SYSTEM AND SPHERE-PLANES

**4.1. Spherical system and sphere-planes of complementary surfaces.** An isothermic surface  $(\Lambda, \eta)$  in  $S^n$  together with a Darboux transform  $\hat{\Lambda}$  define two interesting sphere congruences. First,  $\Lambda, \hat{\Lambda}$  envelop the congruence of 2-spheres given by

$$V_R := \Lambda^{(1)} \oplus \hat{\Lambda} = \Lambda \oplus \hat{\Lambda}^{(1)}.$$

Second, define the *spherical system* of  $\Lambda$  and  $\hat{\Lambda}$  as the  $(n - 2)$ -sphere congruence

$$\mathcal{C} := \Lambda \oplus \hat{\Lambda} \oplus V_R^\perp = (\Lambda^{(1)})^\perp \oplus \hat{\Lambda}.$$

This is the family of codimension 2 spheres that cut both  $\Lambda$  and  $\hat{\Lambda}$  orthogonally at each point. Orthoprojection of  $d$  onto  $\mathcal{C}$  defines a connection  $\mathcal{D}$  which is flat: indeed,  $\Lambda \oplus \hat{\Lambda}$  and  $V_R$  are parallel subbundles on which  $\mathcal{D}$  is flat,  $V_R$  since  $\Lambda$  has flat normal bundle and  $\Lambda \oplus \hat{\Lambda}$  since  $\mathcal{D}$  is also the orthoprojection of  $d + m\eta$  with respect to which  $\hat{\Lambda}$  is parallel by definition. In codimension 1, this flatness amounts to the assertion that  $\mathcal{C}$  defines a *cyclic system* of circles, whence our terminology.

Finally, let  $w \in \mathbb{R}_x^{n+1,1}$  and set

$$\mathcal{P} := \mathcal{C} + \langle w \rangle.$$

On the open set where  $w \notin \mathcal{C}$ ,  $\mathcal{P}$  defines a congruence of hyperspheres we call the *sphere-planes congruence* of  $\Lambda$  and  $\hat{\Lambda}$  with respect to  $w$ . Geometrically,  $\mathcal{P}$  is the unique congruence of totally geodesic hyperspheres in  $E(w)$  containing the spherical system.

Darboux [12] proves that an isothermic surface in  $\mathbb{R}^3$  is special in the classical sense if and only if it admits a pair of possibly complex conjugate Darboux transforms for which the corresponding circle-planes coincide and, in this case, those Darboux transforms are complementary surfaces. This all goes through in arbitrary codimension and for arbitrary space-forms at least in the case of real sphere-planes.

**Theorem 4.1.** *Let  $(\Lambda, \eta)$  be a full isothermic surface in  $S^n$ ,  $n \geq 3$ , with two Darboux transforms  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  of  $(\Lambda, \eta)$  of parameters  $m_1 \neq m_2$ , such that there is a  $w \in \mathbb{R}^{n+1,1} \setminus \Lambda_x^\perp$ , for all  $x$ , which never belongs to the spherical systems of  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  <sup>5</sup>. If the sphere-planes of  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  associated to  $w$  coincide and contain*

<sup>5</sup>that is, the spherical systems of the two Darboux pairs  $\Lambda$  and  $\hat{\Lambda}_1$ , and  $\Lambda$  and  $\hat{\Lambda}_2$ .

no principal direction of  $\Lambda$ , then  $(\Lambda, \eta)$  is a special isothermic surface of type 2, with respect to a polynomial  $p(t)$  with  $p_0 \in \langle w \rangle$ , and such that  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  are complementary surfaces of  $(\Lambda, \eta)$  with respect to  $p(t)$ .

Conversely, if  $(\Lambda, \eta)$  is special isothermic of type 2 in  $E(w)$  with complementary surfaces  $\hat{\Lambda}_1, \hat{\Lambda}_2$  then the sphere-planes through each  $\hat{\Lambda}_i$  and  $w$  coincide.

*Proof.* Let  $F_i$  be a  $(d + m_i\eta)$ -parallel lift of  $\hat{\Lambda}_i$ . We assume that the sphere-planes coincide:

$$\mathcal{P} = \Lambda \oplus \hat{\Lambda}_1 \oplus V_1^\perp \oplus \langle w \rangle = \Lambda \oplus \hat{\Lambda}_2 \oplus V_2^\perp \oplus \langle w \rangle,$$

where  $V_i$  is the enveloped sphere congruence of  $\Lambda$  and  $\hat{\Lambda}_i$ . We can therefore write

$$(11) \quad F_1 = \xi + \beta F_2 + Q$$

where  $\beta$  is a function,  $\xi \in \Gamma(\Lambda^{(1)})^\perp$  and  $Q \in \Gamma\langle w \rangle$ . We are going to prove that  $\beta$  and  $Q$  are constant.

As in (4), for a lift  $F \in \Gamma\Lambda$ , we write

$$\eta = e^{-2\theta} F \wedge (-F_u du + F_v dv),$$

for curvature line coordinates  $u, v$ . Now apply  $\partial/\partial u + m_1\eta_{\partial/\partial u}$  to (11) to get

$$0 = \xi_u + \beta_u F_2 + (m_1 - m_2)\beta\eta_{\partial/\partial u} F_2 + Q_u + m_1\eta_{\partial/\partial u} Q.$$

Let  $\pi$  denote orthoprojection onto  $V_2$  and set  $\pi^\perp = 1 - \pi$ . We apply  $\pi$  to the last equation and rearrange to get:

$$\pi(\xi_u) + (m_1 - m_2)\beta\eta_{\partial/\partial u} F_2 + m_1\eta_{\partial/\partial u} Q = -\beta_u F_2 - Q_u + \pi^\perp(Q_u).$$

Here the left hand side takes values in  $\Lambda_u = \langle F, F_u \rangle$  since  $u, v$  are curvature line coordinates, while the right hand side lies in  $\mathcal{P}$ . Our assumption on principal directions is that  $\Lambda_u \cap \mathcal{P} = \Lambda$  from which we deduce that  $Q_u$  takes values in  $\Lambda \oplus \hat{\Lambda}_2 \oplus V_2^\perp$ . However, this last has trivial intersection with  $\langle w \rangle$  so that  $Q_u = 0$ . Since  $F_2$  does not lie in  $\Lambda$ , we now conclude that  $\beta_u = 0$  also. Similarly, we obtain  $Q_v = 0$  and  $\beta_v = 0$ .

Now let  $p(t) = p_0 + p_1 t + p_2 t^2$  be the polynomial such that

$$p(m_1) = m_1 F_1, \quad p(m_2) = m_2 \beta F_2 \quad \text{and} \quad p(0) = \frac{m_1 m_2}{m_2 - m_1} Q.$$

One readily computes from (11) that  $p_2 = \xi/(m_1 - m_2) \in \Gamma(\Lambda^{(1)})^\perp$  so that  $\eta p_2 = 0$ . Thus  $(d + t\eta)p(t)$  is quadratic in  $t$  with zeros at  $m_1, m_2$  and 0 and so vanishes identically. We conclude that  $p(t)$  is our desired polynomial conserved quantity.

For the converse, if  $\Lambda$  has a polynomial conserved quantity  $p(t) = p_0 + p_1 t + p_2 t^2$  and  $\hat{\Lambda} = \langle p(m) \rangle$  is a complementary surface then the corresponding spherical system is  $\mathcal{C} = (\Lambda^{(1)})^\perp \oplus \langle p(m) \rangle$  so that the sphere-planes are given by

$$\mathcal{C} + \langle w \rangle = (\Lambda^{(1)})^\perp + \langle p_0 + p_1 m + p_2 m^2 \rangle + \langle w \rangle = (\Lambda^{(1)})^\perp + \langle w, p_1 \rangle,$$

which is independent of  $m$ .  $\square$

A similar but simpler argument shows that the stronger condition of coincident spherical systems characterises those special isothermic surfaces of type 1 that admit two complementary surfaces<sup>6</sup>:

**Theorem 4.2.** *Let  $(\Lambda, \eta)$  be a full isothermic surface in  $S^n$ ,  $n \geq 3$ , with two Darboux transforms  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  of  $(\Lambda, \eta)$  of parameters  $m_1 \neq m_2$ . The spherical systems through  $\hat{\Lambda}_1$  and  $\hat{\Lambda}_2$  coincide if and only if  $\Lambda$  is special isothermic of type 1 with the  $\hat{\Lambda}_i$  complementary surfaces.*

<sup>6</sup>In codimension 1, these are the surfaces of constant mean curvature  $H$  in a space-form of sectional curvature  $K$  for which  $H^2 + K > 0$ .

*Proof.* If the spherical systems coincide, we have, with  $F_i$   $(d + m_i\eta)$ -parallel lifts of  $\hat{\Lambda}_i$ ,

$$F_1 = \xi + \beta F_2,$$

for a function  $\beta$  and  $\xi \in \Gamma(\Lambda^{(1)})^\perp$ . Applying  $d + m_1\eta$  to this yields

$$0 = d\xi + d\beta F_2 + (m_1 - m_2)\eta F_2 \equiv d\beta F_2 \pmod{\Lambda^\perp}$$

so that  $\beta$  is constant. Now define  $p(t) = p_0 + p_1 t$  by requiring that

$$p(m_1) = F_1, \quad p(m_2) = \beta F_2.$$

One computes that  $p_1 = \xi/(m_1 - m_2)$  so that  $\eta p_1 = 0$ . Now  $(d + t\eta)p(t)$  is first order in  $t$  with zeros at  $m_1, m_2$  and so vanishes.

The converse is straightforward.  $\square$

**4.2. Enveloping surfaces of circle-planes of complementary surfaces.** Darboux [12, p. 507] (see also [1, §15]) proves that, for a special isothermic surface in  $\mathbb{R}^3$ , the enveloping surface of the congruence of circle-planes of the complementary surfaces is isometric to a quadric in  $\mathbb{R}^{2,1}$ . We now prove a version of this result in arbitrary codimension.

Let  $(\Lambda, \eta)$  be a special isothermic surface of type 2 in  $\mathbb{R}^n = E(w)$ ,  $w \in \mathcal{L}$ , with respect to the polynomial  $p(t) = p_0 + p_1 t + p_2 t^2$ . Assume that  $p_2$  is a unit section and write  $p_0 = Bw$  for a constant  $B$  (possibly zero). Let  $\hat{\Lambda} = \langle p(m) \rangle$  be a complementary surface of  $(\Lambda, \eta)$ .

Contemplate the congruence of circles  $\mathcal{C} = \Lambda \oplus \hat{\Lambda} \oplus \langle p_2 \rangle$  which cuts both  $\Lambda$  and  $\hat{\Lambda}$  orthogonally in the direction defined by  $p_2$ . The congruence  $\mathcal{P} = \mathcal{C} \oplus \langle w \rangle$  of planes through these circles is defined whenever  $w \notin \mathcal{C}$ .

**Theorem 4.3.**

- (1) *The congruence  $\mathcal{P}$  generically admits an enveloping surface  $\Lambda_e$ .*
- (2) *The section of  $\Lambda_e$  which takes values in  $E(w)$  is locally isometric to a quadric in  $\mathbb{R}^{2,1}$ .*

*Proof.* Consider sections of  $\mathcal{P}$  of the form  $G = \alpha p_2 + \beta p(m) + sw = G_1 + sw$  for functions  $\alpha, \beta, s$ . Such a section will span an enveloping surface if it is null and  $dG_1$  takes values in  $\mathcal{P}$ . However,  $dp_2$  is proportional to  $dp(m) = -m\eta p(m)$  modulo  $\Lambda$  (the trace-free second fundamental form of  $\Lambda$  is proportional to the holomorphic quadratic differential  $q$ ) so that  $\alpha, \beta$  can be chosen to ensure that  $dG_1 \in \Omega^1(\mathcal{C})$ . Now  $s$  can be chosen to make  $G$  null so long as  $(G_1, w)$  is non-zero.

In this situation, we normalise so that  $(G, w) = -1$  and let  $\Lambda_e$  be the span of  $G$ . We now show that  $G$  is isometric to a quadric.

Observe that  $p(m)$  is a  $\mathcal{D}$ -parallel section of  $\mathcal{C}$  where  $\mathcal{D}$  is the flat connection on  $\mathcal{C}$  given by the orthoprojection of  $d$  onto  $\mathcal{C}$  (indeed,  $\mathcal{C}$  is a parallel subbundle of the spherical system through  $p(m)$ ). Now construct a parallel frame of  $\mathcal{C}$  by taking  $F \in \Gamma\Lambda$  with  $(F, p(m)) = -1$  and setting  $Z := (p_2, p(m))F + p_2$ : a unit section of  $\mathcal{C}$  orthogonal to  $\Lambda$  and  $\hat{\Lambda}$ . We have

$$G_1 = \alpha p_2 + \beta p(m) = \alpha(Z - (p_2, p(m))F) + \beta p(m) = \alpha Z + \beta p(m) + \gamma F,$$

where

$$(12) \quad \gamma = -\alpha(p_2, p(m)).$$

Since  $(G, w) = -1$ , equivalently,  $(G_1, w) = -1$ , we obtain

$$\alpha(p_2, w) + \beta(p(m), w) = -1,$$

so that

$$\alpha^2(p_2, p_0) + \alpha\beta(p(m), p_0) + \alpha B = 0.$$

Now substitute (12) to eliminate the non-constant inner product  $(p_2, p_0)$  and deduce

$$(13) \quad - (m^2 + m(p_2, p_1))\alpha^2 + (m(p_1, p_0) - m^4 - m^3(p_2, p_1))\alpha\beta \\ - \alpha\gamma - m^2\beta\gamma + B\alpha = 0.$$

Note that all the inner products  $(p_i, p_j)$  occurring in (13) are constant.

Since  $\mathcal{D}$  is flat, there are local gauge transformations  $\Psi : (\mathcal{C}, \mathcal{D}) \cong (\mathbb{R}^{2,1}, d)$  and we see that  $g = \Psi \circ G_1$  has image in the quadric defined by (13). Moreover, since  $(dG_1, w) = 0$  and  $dG_1 \in \Omega^1(\mathcal{C})$ , we get

$$(dG, dG) = (dG_1, dG_1) = (\mathcal{D}G_1, \mathcal{D}G_1) = (dg, dg)$$

so that  $\Lambda_e$ , with the metric induced from  $E(w)$ , is isometric to the quadric (13) wherever it immerses.  $\square$

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